United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

Solutions and investigations

6 November 2018

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some problems for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	C	D	E	B	B	A	B	A	D	B	A	E	D	D	D	B	A	C	E	A	C	B	E	C

1. When the following are evaluated, how many of the answers are odd numbers?

$$1^2,\ 2^3,\ 3^4,\ 4^5,\ 5^6$$

A 1 B 2 C 3 D 4 E 5

Solution **C**

When two odd numbers are multiplied, the result is another odd number. Therefore an odd integer raised to a positive power results in an odd number. Likewise, an even integer raised to a positive power results in an even number.

It follows that, of the given numbers, 1^2, 3^4 and 5^6 are odd numbers, while 2^3 and 4^5 are even.

Therefore 3 of the given expressions evaluate to an odd number.

For investigation

1.1 How many of the numbers n^2, where n takes all the integers values from 1 to 100, inclusive, are odd?

1.2 How many of the numbers n^2, where n takes all the integers values from 1 to 100, inclusive, are

(a) divisible by 3?

(b) divisible by 4?

(c) divisible by 6?

(d) divisible by 12?

1.3 How many of the numbers n^3, where n takes all the integers values from 1 to 100, inclusive, are divisible by 8?

1.4 Prove that an odd positive integer raised to a positive power results in an odd number, and that an even positive integer raised to a positive power results in an even number.

1.5 Let d, n and k be positive integers. Find a criterion for the number n^k to be divisible by d.

2. The positive integer 2018 is the product of two primes.

What is the sum of these two primes?

A 1001 B 1010 C 1011 D 1100 E 1101

Solution **C**

The prime factorization of 2018 is given by $2018 = 2 \times 1009$. The sum of the two prime factors is $2 + 1009 = 1011$.

Note

There is no very quick way to check that 1009 is prime, as this would involve checking that no prime ≤ 29 is a divisor. [Why is this enough?] Fortunately, the question tells us that 2018 is the product of two primes. We can conclude from this that the second factor 1009 is prime.

3. Which of the following shows the digit 6 after it has been rotated clockwise through $135°$?

A  B  C  D  E 

SOLUTION **D**

Because $135° = 90° + 45°$, a rotation through $135°$ clockwise is equivalent to a rotation clockwise through $90°$, which is a quarter turn, followed by a rotation clockwise through $45°$, which is one-eighth of a complete turn.

Therefore, as the diagram shows, a rotation through $135°$ clockwise, results in the configuration given as option D.

4. Which of the following is not a multiple of 5?

A $2019^2 - 2014^2$ B $2019^2 \times 10^2$ C $2020^2 \div 101^2$
D $2010^2 - 2005^2$ E $2015^2 \div 5^2$

SOLUTION **E**

Using the factorization of the difference of two squares, we see that $2019^2 - 2014^2 = (2019 - 2014)(2019 + 2014) = 5 \times 4033$, and hence is a multiple of 5.

$2019^2 \times 10^2$ is a multiple of 10^2, that is, a multiple of 100, and hence is a multiple of 5.

$2020^2 \div 101^2 = (20 \times 101)^2 \div 101^2 = 20^2 = 400$ and so is a multiple of 5.

Because 2010^2 and 2005^2 are both multiples of 5, their difference, $2010^2 - 2005^2$, is also a multiple of 5.

However, $2015^2 \div 5^2 = \left(\frac{2015}{5}\right)^2 = 403^2$ and, because 403 is not a multiple of 5, 403^2 is not a multiple of 5.

Hence the correct answer is option E.

FOR INVESTIGATION

4.1 Prove the following facts about divisibility by 5 that are used in the above solution.

(a) For all positive integers m and n, if m and n are multiples of 5, then $m - n$ is a multiple of 5.

(b) For all positive integers m, if m is not a multiple of 5, then m^2 is not a multiple of 5.

4.2 Which of the statements in 4.1 are true when '5', is replaced by '6'?

4.3 Which of the statements in 4.1 are true when '5', is replaced by '20'?

4.4 For which integers d is it true that for all positive integers m, if m is not a multiple of d, then m^2 is not a multiple of d?

5. Which of the following numbers is the largest?

A $\frac{397}{101}$ B $\frac{487}{121}$ C $\frac{596}{153}$ D $\frac{678}{173}$ E $\frac{796}{203}$

SOLUTION **B**

COMMENTARY

Without the use of a calculator it is not feasible in the time available to answer this question by calculating the values of these fractions to an appropriate number of decimal places.

It would also not be reasonable to decide the relative sizes of, for example, $\frac{397}{101}$ and $\frac{487}{121}$ by using the fact that

$$\frac{397}{101} < \frac{487}{121} \Leftrightarrow 397 \times 121 < 487 \times 101.$$

Instead, the best approach here is to notice that all five fractions are close to 4, and then to decide which of them are less than 4, and which are greater than 4.

We note that

$$\frac{397}{101} < \frac{404}{101} = 4,$$
$$\frac{487}{121} > \frac{484}{121} = 4,$$
$$\frac{596}{153} < \frac{612}{153} = 4,$$
$$\frac{678}{173} < \frac{692}{173} = 4,$$

and

$$\frac{796}{203} < \frac{812}{203} = 4.$$

These calculations show that the fraction given as option B is the only one that is greater than 4, and hence is the largest of the given numbers.

FOR INVESTIGATION

5.1 Which of the following numbers is the smallest?

$$\frac{527}{105}, \quad \frac{617}{123}, \quad \frac{707}{141}, \quad \frac{803}{161}, \quad \frac{917}{183}.$$

6. Which of the following is equal to $25 \times 15 \times 9 \times 5.4 \times 3.24$?

A 3^9 B 3^{10} C 3^{11} D 3^{14} E 3^{17}

Solution **B**

Commentary

The last thing we want to do here is to do the multiplications to work out the value of $25 \times 15 \times 9 \times 5.4 \times 3.24$, and then to factorize the answer.

Instead, we write the decimals as fractions, then factorize the individual numbers, and do some cancellation.

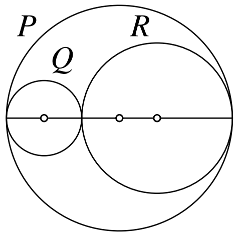
We have $25 = 5^2$, $15 = 3 \times 5$, $9 = 3^2$, $5.4 = \frac{54}{10} = \frac{27}{5} = \frac{3^3}{5}$ and $3.24 = \frac{324}{100} = \frac{81}{25} = \frac{3^4}{5^2}$. Therefore,

$$25 \times 15 \times 9 \times 5.4 \times 3.24 = 5^2 \times (3 \times 5) \times 3^2 \times \frac{3^3}{5} \times \frac{3^4}{5^2}.$$

We can now cancel the factors 5^2 and 5 in the numerator and denominator to deduce that

$$\begin{aligned} 25 \times 15 \times 9 \times 5.4 \times 3.24 &= 3 \times 3^2 \times 3^3 \times 3^4 \\ &= 3^{10}. \end{aligned}$$

7. The circles P, Q and R are all tangent to each other. Their centres all lie on a diameter of P, as shown in the figure.



What is the value of $\dfrac{\text{circumference of } Q + \text{circumference of } R}{\text{circumference of } P}$?

A 1 B $\frac{1}{2}$ C $\frac{1}{3}$ D $\frac{1}{4}$

E more information needed

Solution **A**

Let the radius of the circle Q be q and the radius of the circle R be r. We see that the diameter of the circle P is $2q + 2r$ It follows that the radius of P is $q + r$.

We now use the formula

$$\text{circumference} = 2\pi \times \text{radius},$$

to deduce that

$$\begin{aligned} \frac{\text{circumference of } Q + \text{circumference of } R}{\text{circumference of } P} &= \frac{2\pi q + 2\pi r}{2\pi(q + r)} \\ &= \frac{2\pi(q + r)}{2\pi(q + r)} \\ &= 1 \end{aligned}$$

8. What are the last two digits of 7^{2018}?

A 07 B 49 C 43 D 01 E 18

Solution **B**

Commentary

Clearly, we cannot answer this question by fully evaluating 7^{2018} and then looking to see which are its last two digits.

Instead, we make use of the fact that the last two digits of a product $a \times b$ is determined just by the last two digits of a and b, and then we look for a pattern.

It is convenient to introduce some notation for the last two digits of an integer. There is no standard notation for this (but see 8.1, below). For the purpose of this question we use the notation $[n]$ for the number consisting of the last two digits of the integer n. For example, $[12345] = 45$.

Using this notation we can express the fact we use by the equation

$$[m \times n] = [[m] \times [n]].$$

We then have

$$\begin{aligned}
[7^1] &= [7] = 07 \\
[7^2] &= [49] = 49 \\
[7^3] &= [7^2 \times 7] = [[7^2] \times [7]] = [49 \times 7] = [343] = 43 \\
[7^4] &= [7^3 \times 7] = [[7^3] \times [7]] = [43 \times 7] = [301] = 01 \\
[7^5] &= [7^4 \times 7] = [[7^4] \times [7]] = [01 \times 7] = [7] = 07.
\end{aligned}$$

Each term in the sequence giving the last two digits of 7^n for $n = 1, 2, 3 \ldots$ depends only on the previous term. Hence as 07 has reoccurred we can deduce that the sequence consisting of the values of $[7^n]$ is made up of repeating cycle of length 4, and so is

$$07, \quad 49, \quad 43, \quad 01, \quad 07, \quad 49, \quad 43, \quad 01, \quad \ldots.$$

Because $2018 = 504 \times 4 + 2$ it follows that $[7^{2018}]$ is the second pair of digits in this cycle, namely, 49.

For investigation

8.1 Show that the formula $[m \times n] = [[m] \times [n]]$ is correct.

Note

The number $[m]$ is the remainder when m is divided by 100. If you are familiar with the language of modular arithmetic, you will see that $[m] \equiv m (\mathrm{mod}\ 100)$.

8.2 What are the last two digits of 9^{2018}?

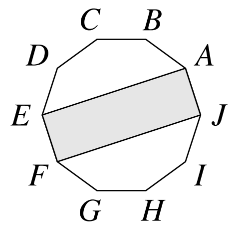
8.3 What are the last two digits of 3^{2018}?

8.4 Prove (by Mathematical Induction) that for every positive integer n, the last two digits of 7^{4n+2} are 49.[To find out about *Mathematical Induction* go to https://nrich.maths.org/4718]

9. The diagram shows a rectangle $AEFJ$ inside a regular decagon $ABCDEFGHIJ$.

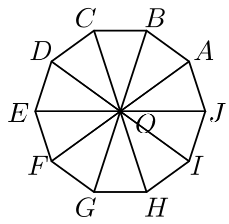
What is the ratio of the area of the rectangle to the area of the decagon?

A $2:5$ B $1:4$ C $3:5$ D $3:10$ E $3:20$



Solution **A**

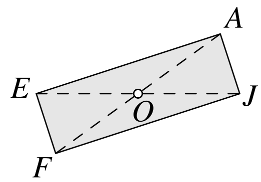
Let O be the centre of the regular decagon. The decagon is divided into ten congruent isosceles triangles by the radii joining O to each of the vertices of the decagon.The triangles AOJ and EOF are two of these ten congruent triangles.



Because O is the centre of the rectangle $AEFJ$ and the diagonals of a rectangle split its area into four equal areas, the triangles EOA and FOJ each have the same area as triangles AOJ and EOF.

Therefore the area of $AEFJ$ is equal to the area of 4 of the 10 congruent triangles that make up the decagon.

Hence the ratio of the area of $AEFJ$ to the area of the decagon $ABCDEFGHI$ is $4:10$ which simplifies to $2:5$.



For investigation

9.1 Prove the following geometrical facts that the above solution tacitly assumes. [A full solution would need to include these proofs.]

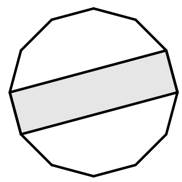
(a) The decagon has a centre, that is, there is circle which goes through all its vertices. (The centre of this circle is the centre of the decagon.)

(b) The centre of the regular decagon is also the centre of the rectangle $AEFJ$.

(c) The triangle EOA has the same area as the triangle AOJ.

9.2 The diagram shows a rectangle whose vertices are two pairs of adjacent vertices of a regular dodecagon.

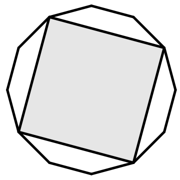
What is the ratio of the area of the rectangle to the area of the dodecagon?



9.3 Generalize the result of the question and the above problem to the case of a rectangle whose vertices are two pairs of adjacent vertices of a regular polygon with $2n$ sides, where n is a positive integer.

9.4 The diagram shows a square inscribed in a regular dodecagon.

What is the ratio of the area of the square to the area of the dodecagon?



10. On a training ride, Laura averages speeds of 12 km/h for 5 minutes, then 15 km/h for 10 minutes and finally 18 km/h for 15 minutes.

What was her average speed over the whole ride?

A 13 km/h B 14 km/h C 15 km/h D 16 km/h E 17 km/h

Solution **D**

To determine Laura's average speed over the whole ride we calculate the total distance she travels, and the total time that she takes.

Because 5 minutes is $\frac{1}{12}$ of an hour, Laura travels $\frac{1}{12} \times 12\,\text{km} = 1\,\text{km}$ when she travels for 5 minutes at 12 km/h.

Because 10 minutes is $\frac{1}{6}$ of an hour, Laura travels $\frac{1}{6} \times 15\,\text{km} = 2.5\,\text{km}$ when she travels for 10 minutes at 15 km/h.

Because 15 minutes is $\frac{1}{4}$ of an hour, Laura travels $\frac{1}{4} \times 18\,\text{km} = 4.5\,\text{km}$ when she travels for 15 minutes at 18 km/h.

Therefore Laura travels at total of $1\,\text{km} + 2.5\,\text{km} + 4.5\,\text{km} = 8\,\text{km}$ in $5 + 10 + 15$ minutes, that is, in 30 minutes, which is half an hour.

Because Laura travels 8 km in half an hour, her average speed is 16 km/h.

For investigation

10.1 Suppose Laura had ridden for a further 20 minutes at 21 km/h. What would then have been her average speed for the whole ride?

10.2 Suppose Laura had extended her training ride by 30 minutes. How fast would she have had to ride in this 30 minutes to make her average speed for the whole hour equal to 20 km/h?

11. How many of the following four equations has a graph that does *not* pass through the origin?

$$y = x^4 + 1 \qquad y = x^4 + x \qquad y = x^4 + x^2 \qquad y = x^4 + x^3$$

A 0 B 1 C 2 D 3 E 4

Solution **B**

The origin has coordinates $(0, 0)$. Thus it is the point where $x = 0$ and $y = 0$. It follows that the graph of an equation passes through the origin if the equation shows that $y = 0$ when $x = 0$.

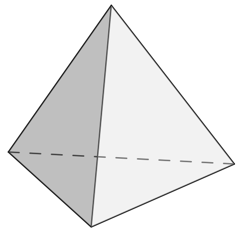
Now, when $x = 0$, the values of $x^4 + 1$, $x^4 + x$, $x^4 + x^2$ and $x^4 + x^3$ are 1, 0, 0 and 0, respectively.

Hence, of the given equations, $y = x^4 + 1$ is the only one whose graph does not pass through the origin.

For investigation

11.1 Sketch the graphs of the four equations given in the question.

12. A regular tetrahedron is a polyhedron with four faces, each of which is an equilateral triangle, as shown. A solid regular tetrahedron is cut into two pieces by a single plane cut.



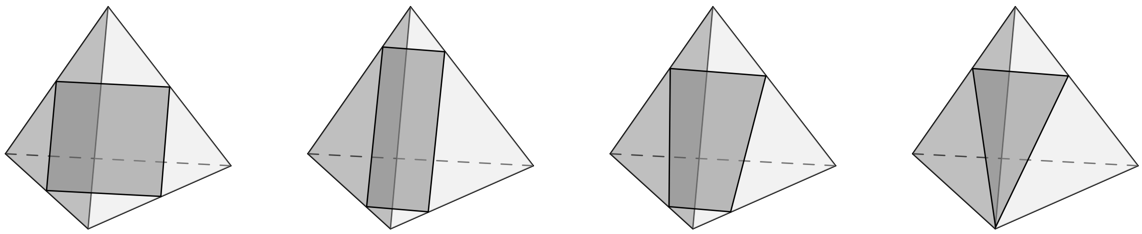
Which of the following could *not* be the shape of the section formed by the cut?

A a pentagon
B a square
C a rectangle that is not a square
D a trapezium
E a triangle that is not equilateral

SOLUTION **A**

When the regular tetrahedron is cut by a single plane cut, each of its four faces is cut at most once. The place where each face is cut becomes the edge of the newly formed section. Therefore a pentagon, with five edges, cannot be formed.

However, each of the other four options is possible, as the following diagrams show.



A square *A rectangle that is not a square* *A trapezium* *A triangle that is not equilateral*

13. The lines $y = x$ and $y = mx - 4$ intersect at the point P.

What is the sum of the positive integer values of m for which the coordinates of P are also positive integers?

A 3 B 5 C 7 D 8 E 10

SOLUTION **E**

At the point P we have both $y = x$ and $y = mx - 4$. Therefore at P we have $x = mx - 4$ and hence $4 = mx - x$. This last equation may be written as $4 = (m - 1)x$. If m and x are both positive integers which satisfy this last equation, then $m - 1$ and x are both positive integers which are factors of 4.

Hence the possible values of $m - 1$ are 1, 2 and 4 with corresponding values of 2, 3 and 5 for m and 4, 2 and 1 for x and y.

Hence the sum of the positive integer values of m for which the coordinates of P are positive integers is $2 + 3 + 5 = 10$.

14. The following twelve integers are written in ascending order:

$$1, x, x, x, y, y, y, y, y, 8, 9, 11.$$

The mean of these twelve integers is 7. What is the median?

A 6 B 7 C 7.5 D 8 E 9

SOLUTION **D**

Because the mean of the given integers is 7, we have

$$1 + x + x + x + y + y + y + y + y + 8 + 9 + 11 = 12 \times 7.$$

We may rewrite this last equation as $1 + 3x + 5y + 28 = 84$ and therefore $3x + 5y = 55$.

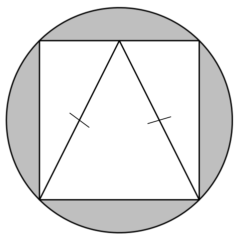
It follows that $3x = 55 - 5y = 5(11 - y)$. Therefore, because x and y are integers, $3x$ is a multiple of 5 and hence x is a multiple of 5.

Because the integers are written in ascending order, $1 \leq x \leq 8$. Hence $x = 5$ and therefore $15 + 5y = 55$. It follows that $5y = 40$ and therefore $y = 8$.

Therefore the twelve integers in ascending order are 1, 5, 5, 5, 8, 8, 8, 8, 8, 9, 11. We now see that the median is 8.

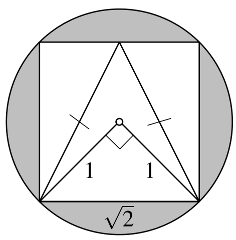
15. A square is inscribed in a circle of radius 1. An isosceles triangle is inscribed in the square as shown.

What is the ratio of the area of this triangle to the area of the shaded region?



A $\pi : \sqrt{2}$ B $\pi : 1$ C $1 : 4$
D $1 : \pi - 2$ E $2 : \pi$

SOLUTION **D**



Let the sides of the square have length s. We see from the diagram that, by Pythagoras' theorem, $s^2 = 1^2 + 1^2$ and hence $s = \sqrt{2}$.

The area of the circle is $\pi \times 1^2$, which equals π. The area of the square is $\sqrt{2} \times \sqrt{2}$, which equals 2. Hence the area of the shaded region is $\pi - 2$.

The base of the isosceles triangle is one of the sides of the square and so has length $\sqrt{2}$. The perpendicular height of the isosceles triangle is also $\sqrt{2}$. The area of the triangle is therefore $\frac{1}{2}(\sqrt{2} \times \sqrt{2})$ which equals 1.

Hence the ratio of the area of the triangle to the area of the shaded region is $1 : \pi - 2$.

FOR INVESTIGATION

15.1 Show that the area of the isosceles triangle is half the area of the square.

16. The numbers p, q, r and s satisfy the following equations:

$$p + 2q + 3r + 4s = k \qquad 4p = 3q = 2r = s.$$

What is the smallest value of k for which p, q, r and s are all positive integers?

A 20 B 24 C 25 D 77 E 154

Solution **D**

From the set of equations $4p = 3q = 2r = s$, we have $q = \frac{4}{3}p, r = 2p$ and $s = 4p$.

We require p and q to be positive integers. Therefore, from the equation $q = \frac{4}{3}p$ we deduce that p has to be an integer which is a multiple of 3. We also note that if p is a positive integer, then so also will be r and s.

It also follows that

$$\begin{aligned} k &= p + 2q + 3r + 4s = p + 2(\tfrac{4}{3}p) + 3(2p) + 4(4p) \\ &= p + \tfrac{8}{3}p + 6p + 16p = \tfrac{77}{3}p. \end{aligned}$$

Because p is a positive integer which is a multiple of 3, its smallest value is 3. Therefore the smallest value of k is $\frac{77}{3} \times 3$, which equals 77.

17. Bethany has 11 pound coins and some 20p coins and some 50p coins in her purse. The mean value of the coins is 52 pence.

Which could not be the number of coins in the purse?

A 35 B 40 C 50 D 65 E 95

Solution **B**

We suppose that numbers of 20p and 50p coins that Bethany has are m and n, respectively. (Note that m and n are positive integers.)

Then Bethany has c coins, where $c = 11+m+n$. The total value of these coins is $11+0.20m+0.50n$ pounds, which equals $1100 + 20m + 50n$ pence.

Because the mean value of Bethany's coins is 52 pence, $\dfrac{1100 + 20m + 50n}{11 + m + n} = 52$. It follows that $1100 + 20m + 50n = 52(11 + m + n)$ and hence $1100 + 20m + 50n = 572 + 52m + 52n$. This equation may be rearranged as $2n = 528 - 32m$, which simplifies to $n = 264 - 16m$.

It follows that $c = 11 + m + n = 11 + m + (264 - 16m) = 275 - 15m$. Therefore $275 - c = 15m$. Thus $275 - c$ is an integer multiple of 15.

We now consider the different values for c given by the options in the question.

We have $275 - 35 = 240 = 15 \times 16$, $275 - 50 = 225 = 15 \times 15$, $275 - 65 = 210 = 15 \times 14$ and $275 - 95 = 180 = 15 \times 12$. Hence Bethany could have 35 or 50 or 65 or 95 coins in her purse.

However $275 - 40 = 235$ which is not an integer multiple of 15. Hence Bethany could not have 40 coins in her purse.

18. P, Q and R are the three angles of a triangle, when each has been rounded to the nearest degree.

Which of the following is the complete list of possible values of $P + Q + R$?

A $179°$, $180°$ or $181°$ B $180°$, $181°$ or $182°$ C $178°$, $179°$ or $180°$
D $180°$ E $178°$, $179°$, $180°$, $181°$ or $182°$

Solution **A**

We suppose that the actual angles of the triangle are P', Q' and R', which are rounded to P, Q and R, respectively.

The sum of the angles of a triangle is $180°$ and therefore

$$P' + Q' + R' = 180°.$$

When an angle is rounded up to the nearest degree, it is increased by at most $0.5°$; when it is rounded down to the nearest degree, it is decreased by at most $0.5°$. Therefore,

$$\begin{aligned} P' - 0.5° &\leq P \leq P' + 0.5°, \\ Q' - 0.5° &\leq Q \leq Q' + 0.5°, \\ \text{and} \quad R' - 0.5° &\leq R \leq R' + 0.5°. \end{aligned}$$

Adding these inequalities gives

$$P' + Q' + R' - 1.5° \leq P + Q + R \leq P' + Q' + R' + 1.5°.$$

Therefore, as $P' + Q' + R' = 180°$,

$$178.5° \leq P + Q + R \leq 181.5°.$$

Each of P, Q and R is an integer number of degrees. Hence $P + Q + R$ is also an integer number of degrees. It follows that the only possible values of $P + Q + R$ are $179°$, $180°$ and $181°$.

To complete the solution we show that each of these possible values actually occurs for some triangle. This is shown by the following examples.

$P' = 60.3°$, $Q' = 60.3°$, $R' = 59.4°$ gives $P = 60°$, $Q = 60°$, $R = 59°$. Hence $P + Q + R = 179°$.

$P' = 60°$, $Q' = 60°$, $R' = 60°$ gives $P = 60°$, $Q = 60°$, $R = 60°$. Hence $P + Q + R = 180°$.

$P' = 59.7°$, $Q' = 59.7°$, $R' = 60.6°$ gives $P = 60°$, $Q = 60°$, $R = 61°$. Hence $P + Q + R = 181°$.

We deduce that $179°$, $180°$, $181°$ is a complete list of the possible values of $P + Q + R$.

For investigation

18.1 Give an example of a triangle whose angles are all different non-integer numbers of degrees, but whose rounded angles have sum $180°$.

18.2 P, Q, R and S are the four angles of a quadrilateral, when each has been rounded to the nearest degree. Give a list of all the values that $P + Q + R + S$ can take.

18.3 The angles of a polygon with n vertices are each rounded to the nearest integer. Give, in terms of n, a list of the values that the sum of these rounded angles can take.

19. How many pairs of numbers (m, n) are there such that the following statement is true?

'A regular m-sided polygon has an exterior angle of size $n°$ and
a regular n-sided polygon has an exterior angle of size $m°$.'

A 24 B 22 C 20 D 18 E 16

Solution **C**

We first note that a polygon has at least 3 sides, so we need consider only cases where $m \geq 3$ and $n \geq 3$.

The exterior angle of a regular m-sided polygon is $\left(\frac{360}{m}\right)^{\circ}$. Hence, the condition for a regular m-sided polygon to have an exterior angle of $n°$ is that $n = \frac{360}{m}$. This condition may be rewritten as $mn = 360$.

Similarly, this is the condition that a regular n-sided polygon has an exterior angle of $m°$.

Therefore the pair of numbers (m, n) satisfies the condition of the question if, and only if m and n are both at least 3 and $mn = 360$.

There are 10 ways of expressing 360 as the product of positive integers which are both at least 3. These are

$$3 \times 120,\ 4 \times 90,\ 5 \times 72,\ 6 \times 60,\ 8 \times 45,\ 9 \times 40,\ 10 \times 36,\ 12 \times 30,\ 15 \times 24 \text{ and } 18 \times 20.$$

Each of these 10 factorizations can be taken in either order to give two pairs (m, n) which meet the required condition. For example, corresponding to the factorization 3×120 we see that (m, n) may be either $(3, 120)$ or $(120, 3)$.

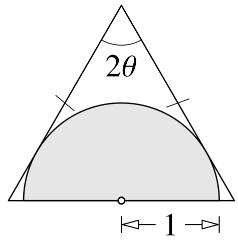
It follows that there are 10×2, that is, 20 pairs of numbers (m, n) meeting the required condition.

For investigation

19.1 Prove that the exterior angle of a regular polygon with n sides is $\left(\frac{360}{n}\right)^{\circ}$.

19.2 For how many different values of n is there a regular polygon with n sides whose interior angles are each an integer number of degrees?

20. The diagram shows a semicircle of radius 1 inside an isosceles triangle. The diameter of the semicircle lies along the 'base' of the triangle, and the angle of the triangle opposite the 'base' is equal to 2θ. Each of the two equal sides of the triangle is tangent to the semicircle.



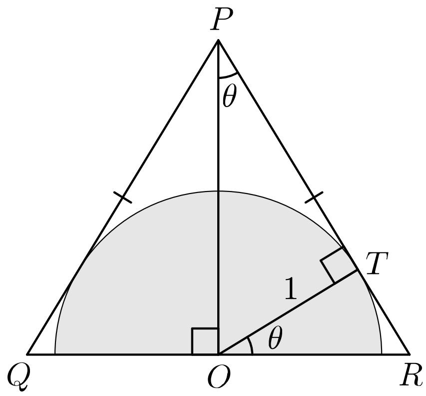
What is the area of the triangle?

A $\frac{1}{2}\tan 2\theta$ B $\sin\theta\cos\theta$ C $\sin\theta + \cos\theta$

D $\frac{1}{2}\cos 2\theta$ E $\dfrac{1}{\sin\theta\cos\theta}$

Solution **E**

Let the vertices of the triangle be P, Q and R, with $PQ = PR$, as shown in the diagram, and let O be the midpoint of QR. We also let T be the point where the semicircle touches PR. In the triangles PQO and PRO, we have $PQ = PR$, $QO = RO$ and the side PO is common. Therefore the triangles are congruent (SSS). Hence the area of the triangle PQR is twice the area of the triangle POR.



It also follows that $\angle QPO = \angle RPO = \theta$. Hence PO is the bisector of $\angle QPR$. Hence O is equidistant from PQ and PR and is therefore the centre of the semicircle.

We can also deduce that $\angle POQ = \angle POR = 90°$.

OT is a radius of the semicircle and therefore has length 1 and is perpendicular to the tangent PR. Hence PTO is a right-angled triangle and therefore

$$\frac{OT}{OP} = \sin\theta$$

and hence

$$OP = \frac{OT}{\sin\theta} = \frac{1}{\sin\theta}.$$

Also, from the triangle POR, we have

$$\frac{OP}{PR} = \cos\theta$$

and hence

$$PR = \frac{OP}{\cos\theta} = \frac{1}{\sin\theta\cos\theta.}$$

If we regard PR as the base of the triangle POR, and OT as its base, we see that

$$\text{area of } POR = \tfrac{1}{2}(PR \times OT) = \frac{1}{2\sin\theta\cos\theta}.$$

It follows that the area of the triangle PQR is $\dfrac{1}{\sin\theta\cos\theta}$.

21. The graph of $y = \dfrac{1}{x}$ is reflected in the line $y = 1$. The resulting image is reflected in the line $y = -x$.

What is the equation of the final graph?

A $y = \dfrac{-1}{(x+2)}$ B $y = \dfrac{1}{(x-1)}$ C $y = \dfrac{1}{(x-2)}$ D $y = \dfrac{-1}{(x-1)}$ E $y = \dfrac{-1}{(x-2)}$

SOLUTION **A**

COMMENTARY

We give two methods for answering this question.

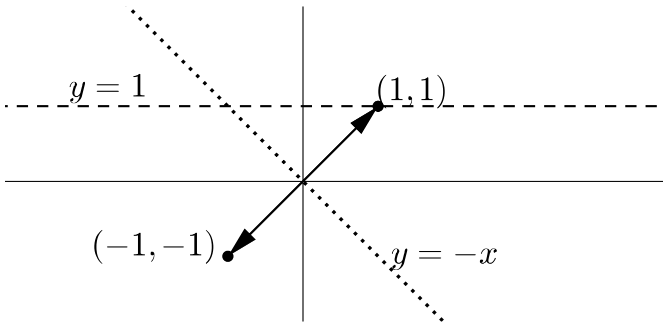
The first quick method is adequate in the context of the SMC where you can assume that one of the options is correct, and you are not required to justify the answer.

The second method uses algebra to calculate the equation of the final graph, and is an example of the kind of answer that you would need to give when fully explained solutions are required.

METHOD 1

When $x = 1$, we have $\dfrac{1}{x} = 1$, and therefore the point with coordinates $(1, 1)$ lies on the graph of $y = \dfrac{1}{x}$.

The point $(1, 1)$ remains fixed when reflected in the line $y = 1$, and then is mapped to the point $(-1, -1)$ after reflection in the line $y = -x$.



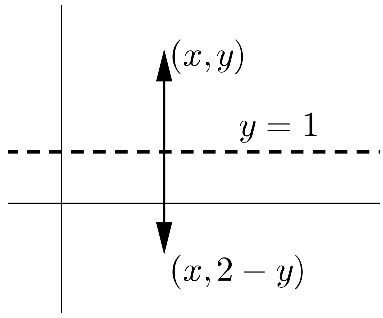
It follows that the point $(-1, -1)$ lies on the final graph. It is easy to check that the equation of option A gives $y = -1$ when $x = -1$, but that none of the other equations have this property. We can therefore eliminate options B, C, D and E, and hence conclude that the correct option is A.

NOTE

This argument conclusively eliminates the options B, C, D and E but does not *prove* that the equation of option A is the equation of the final graph. It only shows that this is a possibility.

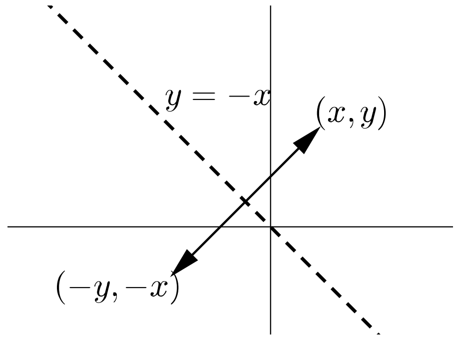
Method 2

Reflection in the line $y = 1$ interchanges the points with coordinates (x, y) and $(x, 2 - y)$, as shown in the first diagram on the right.



Therefore the image of the graph of $y = \dfrac{1}{x}$ after this reflection is the graph of $2 - y = \dfrac{1}{x}$.

Reflection in the line $y = -x$ interchanges the points with coordinates (x, y) and $(-y, -x)$, as shown in the second diagram on the right.



Therefore the image of the graph of $2 - y = \dfrac{1}{x}$ after this reflection is the graph of $2 - (-x) = \dfrac{1}{-y}$. This last equation may be rearranged as $y = \dfrac{-1}{(x + 2)}$.

For investigation

21.1 Check that none of the graphs of $y = \dfrac{1}{(x - 1)}$, $y = \dfrac{1}{(x - 2)}$, $y = \dfrac{-1}{(x - 1)}$ and $y = \dfrac{-1}{(x - 2)}$ goes through the point with coordinates $(-1, -1)$.

21.2 (a) Sketch the graphs of $y = \dfrac{1}{x}$ and $y = \dfrac{-1}{(x + 2)}$.

(b) Indicate geometrically how reflecting the graph of $y = \dfrac{1}{x}$ in the line $y = 1$ and then reflecting the resulting graph in the line $y = -x$ produces the graph of $y = \dfrac{-1}{x + 2}$.

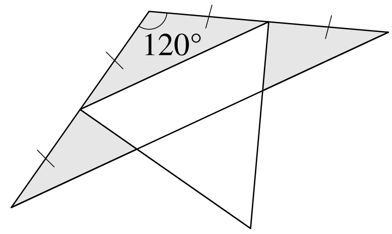
21.3 (a) Show that the image of the point with coordinates (h, k) in the line $y = 1$ is the point with coordinates $(h, 2 - k)$

(b) Let f be some function of x. Show that the image of the graph of $y = f(x)$ after reflection in the line $y = 1$ is the graph of $y = 2 - f(x)$.

21.4 (a) Show that the image of the point with coordinates (h, k) in the line $y = -x$ is the point with coordinates $(-k, -h)$

(b) Let f be some function of x. Show that the image of the graph of $y = f(x)$ after reflection in the line $y = -x$ is the curve whose equation is $x = -f(-y)$.

22. The diagram shows two overlapping triangles; an isosceles triangle with an angle of $120°$ and an equilateral triangle with area 36. Two of the vertices of the equilateral triangle are midpoints of the equal sides of the isosceles triangle.



What is the total area of the shaded regions (inside the isosceles triangle but outside the equilateral triangle)?

A 24 B 26 C 28 D 30 E 32

Solution **C**

Commentary

This problem may be tackled in many different ways. We first give a full solution. We then sketch two more solutions, leaving the details to the reader. You may well find other methods.

Method 1

We let the vertices of the isosceles triangle with an angle of $120°$ be P, Q and R, and those of the equilateral triangle be K, L and M, as shown in the diagram below.

We also let N be the midpoint of KM and S and T be the points where QR meets the lines KL and ML, respectively.

We also suppose that the equilateral triangle KLM has sides of length $2s$

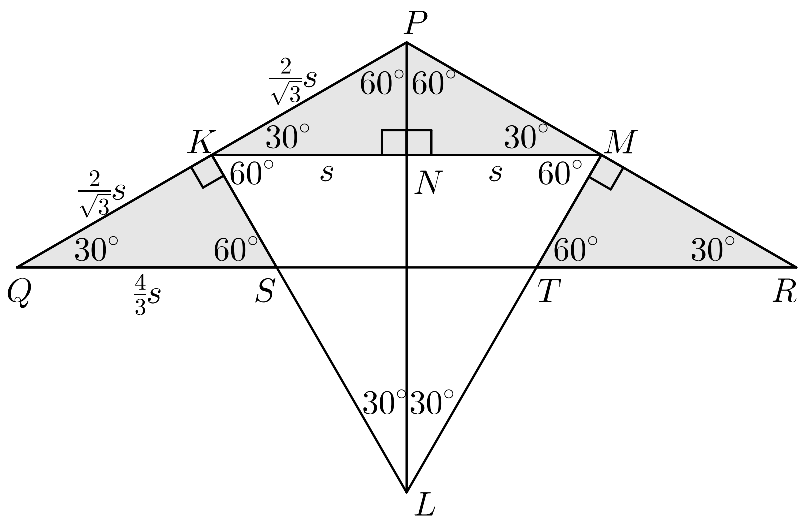


Figure 1

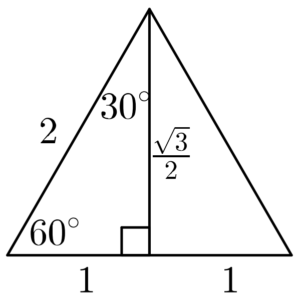
The triangle PQR is isosceles with $PQ = PR$, and K and M are the midpoints of PQ and PR, respectively. Therefore $PK = PM$. Since N is the midpoint of KM we also have $KN = NM$. Therefore the triangles PKN and PMN have sides of the same lengths. Hence they are congruent. Therefore the angles of both these triangles are $90°$, $60°$ and $30°$, as shown in the diagram.

Similarly, the triangles KNL and MNL have sides of the same lengths, and so are congruent. It follows that they also have angles of 90°, 60° and 30° as shown.

Because the triangle PQR is isosceles with $PQ = PR$, it follows that $\angle PQR = \angle PRQ$. Therefore, as $\angle QPR = 120°$ and the sum of the angles in a triangle is 180°, it follows that $\angle PQR = \angle PRQ = 30°$.

The angles on the line PQ at K have sum 180°. It follows that $\angle QKS = 90°$. Therefore QKS is yet another triangle with angles 90°, 60° and 30°. Similarly, the same is true of the triangle RMT. Because $QK = RM$, it follows that the triangles QKS and RMT are congruent.

We have therefore seen that KPN, QSK and LKN are similar triangles with angles 90°, 60° and 30°.



By considering a triangle with these angles as half of an equilateral triangle, we see that in such a triangle the hypotenuse and other two sides are in the ratio $2 : \frac{\sqrt{3}}{2} : 1$.

Therefore, as $KN = s$, it follows that $KP = \frac{2}{\sqrt{3}}s$.

Hence $QK = KP = \frac{2}{\sqrt{3}}s$ and therefore $QS = \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}}s = \frac{4}{3}s$.

The ratio of the areas of similar triangles equals the ratio of the squares of their linear dimensions.

We have seen that the similar triangles KPN, QSK and LKN have hypotenuses of lengths $\frac{2}{\sqrt{3}}s$, $\frac{4}{3}s$ and $2s$, respectively. These are in the ratio $\frac{2}{\sqrt{3}} : \frac{4}{3} : 2$. Hence the areas of these triangles are in the ratio $\frac{4}{3} : \frac{16}{9} : 4$, which is equivalent to $\frac{1}{3} : \frac{4}{9} : 1$.

Therefore the areas of the triangles KPN and QSK are, respectively, $\frac{1}{3}$ and $\frac{4}{9}$ of the area of the triangle LKN. Now $\frac{1}{3} + \frac{4}{9} = \frac{7}{9}$. Hence the sum of the areas of the triangles KPN and QSK is $\frac{7}{9}$ of the area of the triangle LKN.

Similarly the sum of the areas of the triangles MPN and TRM is $\frac{7}{9}$ of the area of the triangle LMN.

It follows that the area of the shaded region is $\frac{7}{9}$ of the area of the equilateral triangle. Hence, the shaded area is $\frac{7}{9} \times 36 = 28$.

Method 2

Let O be the midpoint of QR, and join O to K, L and M.

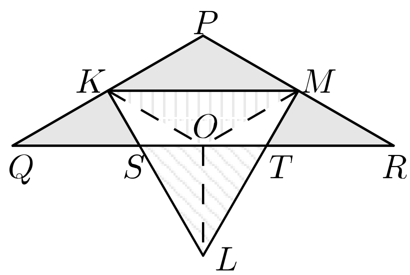


Figure 2

It can be checked that the triangles PKM, KOM, LOK and MOL are all congruent. It follows that the area of the triangle PKM is one third of the area of KLM, and hence the triangle PKM has area 12.

It can also be proved that the length of ST is $\frac{2}{3}$ of the length of KM. Hence the area of the triangle LTS is $\frac{4}{9}$ of the area of KLM, that is 16.

It can also be checked that the triangles KQS, MTR, SLO and LTO are congruent.

Therefore the sum of the areas of the triangles KQS and MTR is equal to the area of the triangle STL, that is, 16.

Hence the shaded area is 12 + 16 = 28.

Method 3

The third solution we give is more visual. It relies on properties of the diagrams which would need detailed justification if a full solution were required.

Consider the diagram in Figure 3, below, which shows the original shape, outlined by heavy lines, together with two copies of it, rotated by $\pm 120°$.

This shows that the hatched area in Figure 3 is equal to one third of the original equilateral triangle, that is, 12. We also see that the large equilateral triangle has area $4 \times 36 = 144$.

Now consider Figure 4, which shows the same diagram, superimposed by several smaller equilateral triangles.

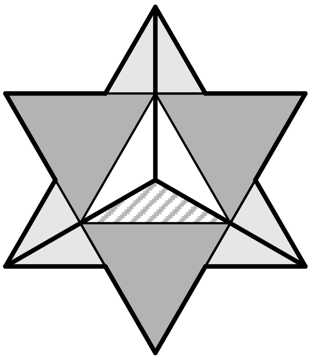


Figure 3

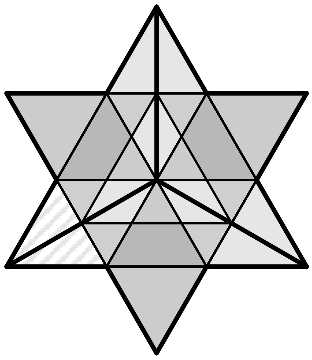


Figure 4

This shows that the hatched area in Figure 4 is equal to one ninth of the large equilateral triangle, that is, 16.

Hence the shaded area in the original figure is 12 + 16 = 28.

For investigation

22.1 Explain why it follows from the fact that in Figure 1 the triangles PKN and PMN are congruent that the angles in these triangles are 90°, 60° and 30°.

22.2 Show that if in a right-angled triangle the hypotenuse has length 2 and one of the other sides has length 1, then the third side has length $\sqrt{3}$.

22.3 Explain why the ratio of the areas of similar triangles equals the ratio of the squares of their linear dimensions.

22.4 Find the area of the rectangle $STMK$ in Figure 1 on the previous page.

22.5 Prove that in Figure 1 the triangles PQL and PRL are equilateral and hence that $PQLR$ is a rhombus.

22.6 Fill in the missing details in the solutions of *Method* 2 and *Method* 3.

23. For particular real numbers a and b, the function f is defined by $f(x) = ax + b$, and is such that $f\big(f\big(f(x)\big)\big) = 27x - 52$.

Which of the following formulas defines the function g such that, for all values of x, $g\big(f(x)\big) = x$?

A $\frac{1}{3}x - 4$ B $\frac{1}{3}x + \frac{4}{3}$ C $4x - 3$ D $\frac{1}{3}x - \frac{4}{3}$ E $3x - 4$

Solution **B**

Because $f(x) = ax + b$, we have that, for all real numbers x,

$$\begin{aligned} f(f(x)) &= f(ax + b) \\ &= a(ax + b) + b \\ &= a^2x + ab + b. \end{aligned}$$

It follows that, for all real numbers x,

$$\begin{aligned} f(f(f(x))) &= f(a^2x + ab + b) \\ &= a(a^2x + ab + b) + b \\ &= a^3x + a^2b + ab + b \\ &= a^3x + (a^2 + a + 1)b. \end{aligned}$$

Because the expressions $a^3x + (a^2 + a + 1)b$ and $27x - 52$ have the same value for all real numbers x, their coefficients match. Hence $a^3 = 27$ and $(a^2 + a + 1)b = -52$. It follows that $a = 3$ and hence $13b = -52$, giving $b = -4$. Therefore the function f is defined by $f(x) = 3x - 4$.

> Commentary
>
> We now need to find the formula for the function g such that $g(f(x)) = x$. If $f(x) = y$ this implies that $g(y) = x$. Hence the method we use is to start with the equation $y = f(x)$ and then rearrange this so that it has the form $x = g(y)$.

We have

$$\begin{aligned} y = f(x) &\Leftrightarrow y = 3x - 4 \\ &\Leftrightarrow 3x = y + 4 \\ &\Leftrightarrow x = \tfrac{1}{3}y + \tfrac{4}{3}. \end{aligned}$$

It follows that g is given by the formula $g(y) = \frac{1}{3}y + \frac{4}{3}$. Rewriting this formula in terms of x gives $g(x) = \frac{1}{3}x + \frac{4}{3}$.

For investigation

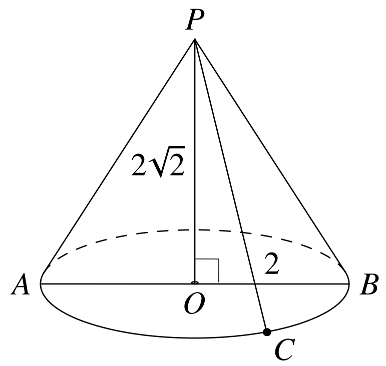
23.1 In each of the following cases find a formula for the function g such that $g(f(x)) = x$.

(a) $f(x) = 7x - 28$.

(b) $f(x) = \sqrt{x + 3} + 5$.

(c) $f(x) = \dfrac{x}{x + 1}$.

24. The diagram shows a circle with centre O which lies in a horizontal plane. The diameter AB has length 4. Point P lies vertically above O and $PO = 2\sqrt{2}$. Point C lies on the semicircular arc AB such that the ratio of the lengths of the arcs AC and CB is 2 : 1.

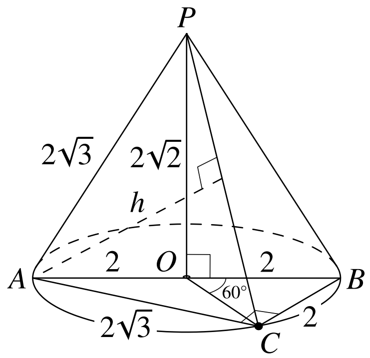


What is the shortest distance from A to PC?

A $\sqrt{2}$ B $\sqrt{3}$ C 2 D $2\sqrt{2}$ E 3

Solution **E**

The triangle POA has a right angle at O. Therefore, by Pythagoras's Theorem, $PA^2 = AO^2 + PO^2 = 2^2 + (2\sqrt{2})^2 = 4 + 8 = 12$. It follows that $PA = \sqrt{12} = 2\sqrt{3}$. Similarly, $PC = 2\sqrt{3}$.



Since arc AC : arc CB = 2 : 1, it follows that $\angle AOC : \angle COB = 2 : 1$. Hence $\angle COB = 60°$.

Now OC and OB both have length 2 as they are radii of the circle with centre O. Since $\angle COB = 60°$, it follows that the triangle COB is equilateral. Hence $BC = 2$.

We now turn our attention to the triangle ACB. In this triangle $\angle ACB = 90°$ because it is the angle in a semicircle, $AB = 4$ and $BC = 2$. Therefore, by Pythagoras' Theorem, $AB^2 = AC^2 + BC^2$ and hence $4^2 = AC^2 + 2^2$. It follows that $AC^2 = 4^2 - 2^2 = 16 - 4 = 12$. Therefore $AC = \sqrt{12} = 2\sqrt{3}$.

We have therefore shown that PAC is an equilateral triangle in which each side has length $2\sqrt{3}$.

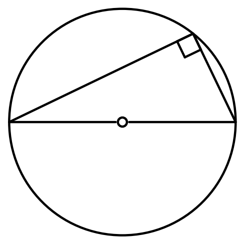
The shortest distance from A to PC is the length, say h, of the perpendicular from A to PC. Thus h is the height of the equilateral triangle PAC.

We have seen in the solution to Question 22 that the height of an equilateral triangle with side length x is $\frac{\sqrt{3}}{2}x$. It follows that

$$h = \frac{\sqrt{3}}{2} \times 2\sqrt{3} = 3.$$

For investigation

24.1 Prove that the angle in a semicircle is a right angle.



Note

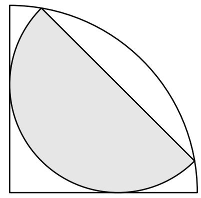
The theorem immediately above is attributed to Thales of Miletus (c624 BCE - c547 BCE), although, as none of his writings has survived, we cannot be sure that he proved this result. For more information about Thales go to the MacTutor History of Mathematics archive:

http://www-history.mcs.st-and.ac.uk

25. A semicircle is inscribed in a quarter circle as shown.

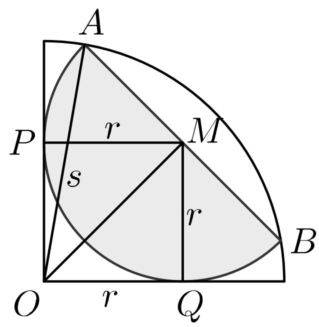
What fraction of the quarter circle is shaded?

A $\frac{1}{3}$ B $\frac{1}{\sqrt{3}}$ C $\frac{2}{3}$ D $\frac{\sqrt{3}}{2}$ E $\frac{1}{\sqrt{2}}$



Solution **C**

Let O be the centre of the quarter circle, A and B be the ends of the diameter of the semicircle and M be the centre of the semicircle, and hence the midpoint of AB.



Let P and Q be the points on the straight edges of the quarter circle where the quarter circle is tangent to the semicircle.

We let the radius of the quarter circle be s and let the radius of the semicircle be r.

Then $OA = s$.

As a tangent to a circle is perpendicular to its radius, $\angle OPM = \angle OQM = 90°$. Because also $\angle POQ = 90°$, all the angle of the quadrilateral $OQMP$ are right angles. Also $PM = r = QM$. Hence $OPMQ$ is a square with side length r.

Applying Pythagoras' theorem to the right-angled triangle OQM gives $OM^2 = r^2 + r^2 = 2r^2$.

The triangle AMO has a right angle at M. Therefore, by Pythagoras' theorem applied to this triangle, we have

$$AO^2 = OM^2 + MA^2$$

and hence

$$\begin{aligned} s^2 &= 2r^2 + r^2 \\ &= 3r^2. \end{aligned}$$

The area of the quarter circle is $\frac{1}{4}\pi s^2$. The area of the shaded semicircle is $\frac{1}{2}\pi r^2$.

Therefore, the fraction of the quarter circle which is shaded is given by

$$\frac{\frac{1}{2}\pi r^2}{\frac{1}{4}\pi s^2} = \frac{2r^2}{s^2} = \frac{2r^2}{3r^2} = \frac{2}{3}.$$

For investigation

25.1 Prove that the triangle AMO has a right angle at M.

25.2 What is the ratio of the perimeter of the quarter circle to the perimeter of the half circle? [In each case the perimeter includes the straight edges.]